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A TWO-SPHERES PROBLEM ON HOMOGENEOUS TREES

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We show that, given m, n two relatively prime natural numbers, if a complex valued function f on a homogeneous tree satisfies the mean value property for all spheres of radius m and all spheres of radius n, then f is harmonic.

1. Introduction

A *homogeneous tree of degree* k ($k\in\mathbf{N}$) is a connected graph without circuit (a tree) such that every vertex has exactly k neighbours. If x and y are two vertices of any tree, there exists a unique sequence $z_0,\dots,z_n$ of vertices all distinct such that z_i is a neighbour of z_{i+1} for $i=0,\dots,n-1$ and $z_0=x$, $z_n=y$; letting $d(x,y)=n$ defines a distance on the tree (see [1] §1.1). Then, up to isometry, there is only one homogeneous tree of degree $k\in\mathbf{N}$: we denote it by $\mathcal{A}_k$; it is infinite as soon as $k\geq 2$, which we will suppose in the following. (For example, $\mathcal{A}_2\simeq\mathbf{Z}$.)

If x is a vertex of $\mathcal{A}_k$ and r a positive integer, we write $S(x,r)$ for the sphere of centre x and radius r, i.e. the set $\{y\in\mathcal{A}_k|d(x,y)=r\}$. The definition of $\mathcal{A}_k$ forces $S(x,1)$ to have k elements for every vertex x. It can be shown by induction that $\operatorname{card}S(x,r)=k(k-1)^{r-1}$ if $r\geq 1$, for every vertex x. Let r be an integer ≥ 0; we define linear operators Σ_r and $\mathcal{M}_r$ on the vector space $\mathcal{F}(\mathcal{A}_k)$ of complex valued functions on $\mathcal{A}_k$ by, if $x\in\mathcal{A}_k$,

$$(\Sigma_r f)(x)=\sum_{y\in S(x,r)} f(y),$$

$$(\mathcal{M}_r f)(x)=\frac{1}{\operatorname{card}S(x,r)}\sum_{y\in S(x,r)} f(y).$$

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A function f on $\mathcal{A}_k$ is said to be *harmonic* if $\mathcal{M}_1 f = f$.

It is easily seen that if f in $\mathcal{F}(\mathcal{A}_k)$ is harmonic, it verifies the *mean value property*: $\mathcal{M}_r f = f$, for every $r \geq 0$ (see §2). By analogy with what occurs on $\mathbf{R}^n$ (Delsarte's theorem, see [3], [6]), we are interested in knowing when the mean value property for *two* values of r implies the harmonicity of f. We have obtained the following partial answer:

Proposition. *Let m, n two distinct natural numbers. Are equivalent:*

i) *for all $k \geq 2$ we have: $f \in \mathcal{F}(\mathcal{A}_k)$ and $\mathcal{M}_m f = f = \mathcal{M}_n f$ imply f is harmonic;*

ii) *m and n are relatively prime.*

Remark. We have followed for the start of our proof the approach of Cohen and Picardello [2], who fully answer the question: To what extent does the property of $\mathcal{M}_r f$ being zero for two values of r imply that f is zero?

2. Preliminaries

For a function f on $\mathcal{A}_k$ and $x \in \mathcal{A}_k$ we have (see [2, p.75])

$$\Sigma_1(\Sigma_1 f)(x) = (\Sigma_2 f)(x) + k(\Sigma_0 f)(x),$$

and if $r \geq 2$,

$$\Sigma_r(\Sigma_1 f)(x) = (\Sigma_{r+1} f)(x) + (k-1)(\Sigma_{r-1} f)(x).$$

Consequently we introduce the polynomials $p_r, q_r \in \mathbf{Z}[X]$ defined for all $r \in \mathbf{Z}_+$ by induction:

$$p_0 = 1,\ p_1 = X,\ p_2 = X^2 - k \ \text{ and } \ p_{r+1} = X p_r - (k-1) p_{r-1} \ \text{ if } r \geq 2;$$

$$q_0 = 1,\ q_1 = X - k \ \text{ and } q_r = p_r - k(k-1)^{r-1} \ \text{ if } r \geq 2.$$

We have $\Sigma_r = p_r(\Sigma_1)$ for any $r \geq 0$. So $\mathcal{M}_r f = f$ is equivalent to $p_r(\Sigma_1)(f) = k(k-1)^r f$, that is, to $q_r(\Sigma_1)(f) = 0$. One can of course define q_r by a direct induction:

$$\begin{cases} q_0 = 1,\ q_1 = X - k,\ q_2 = X^2 - k^2 \ \text{ and} \\ q_{r+1} = X q_r - (k-1) q_{r-1} + (X-k)k(k-1)^{r-1} \ \text{ if } r \geq 2. \end{cases}$$

It follows, by an easy recurrence, that $q_r(k) = 0$, and so $(X-k) | q_r$ for all $r \in \mathbf{N}$.

If $f \in \mathcal{F}(\mathcal{A}_k)$ is harmonic, $(\Sigma_1 - k)(f) = 0$; therefore $q_r(\Sigma_1)(f) = 0$, which shows that f verifies the mean value property on every sphere in $\mathcal{A}_k$.

3. Proof

The implication $i) \Rightarrow ii)$ is almost trivial. For $k = 2$, if m and n are not relatively prime, set $l = \gcd(m, n)$ and let f on $\mathcal{A}_2 \simeq \mathbf{Z}$ be the indicator function of $l\mathbf{Z}$; then $\mathcal{M}_m f = f = \mathcal{M}_n f$ but f is not harmonic.

We now show $ii)\Rightarrow i)$. Suppose that, given $m,n\in\mathbf{N}$ distinct, the polynomials q_n and q_m have $(X-k)$ as only common divisor. By Bézout, there exist $a,b\in\mathbf{Z}[X]$ such that $aq_n+bq_m=X-k$. It follows that if $f\in\mathcal{F}(\mathcal{A}_k)$ satisfies $\mathcal{M}_nf=f=\mathcal{M}_mf$, that is, $q_n(\Sigma_1)(f)=0=q_m(\Sigma_1)(f)$, then $(\Sigma_1-k)(f)=0$, which means f is harmonic.

So we are brought down to study when q_n and q_m have $(X-k)$ as greatest common divisor; in other words, if we define polynomials $Q_n=q_n/(X-k)\in\mathbf{Z}[X]$, we want to find for which $m,n\in\mathbf{N}$ the greatest common divisor of Q_n and Q_m is 1 or, equivalently, the *resultant* of Q_n and Q_m, $\operatorname{res}(Q_n,Q_m)$, is not zero.

Let us briefly recall some facts about the resultant of two polynomials f, g in $K[t]$ (K a field) which are not both constant:

(a) $\operatorname{res}(f,g)$ is the determinant of a matrix, each of whose coefficients is either zero or a coefficient of f or a coefficient of g;

(b) $\operatorname{res}(f,c)=\operatorname{res}(c,f)=c^{\deg f}$ if $c\in K$ and $\deg f>0$;

(c) $\operatorname{res}(f,g)=(-1)^{\deg f\cdot\deg g}\operatorname{res}(g,f)$;

(d) $\operatorname{res}(h\cdot l,g)=\operatorname{res}(h,g)\cdot\operatorname{res}(l,g)$;

(e) $\operatorname{res}(q\cdot g+r,g)=\gamma^{j-l}(-1)^{k(j-l)}\operatorname{res}(r,g)$, where $j=\deg(q\cdot g+r)$, $k=\deg g$, $l=\deg r$ and γ is the leading coefficient of g.

(Here by convention $\deg c=0$ for all $c\in K$.) It follows that $\operatorname{res}(f,g)=0$ if and only if f and g have a common factor in $K[t]$. (See also [4, p.309].)

The Q_r are given by: $Q_1=1$, $Q_2=X+k$ and

$$(*)\qquad Q_{r+1}=XQ_r-(k-1)Q_{r-1}+k(k-1)^{r-1}\quad\text{if}\quad r\geq 2.$$

Hence $\operatorname{res}(Q_m,Q_n)$ is a polynomial in k with integer coefficients (by (a)). But a polynomial in one variable with integer coefficients whose constant term is 1 or -1 never vanishes on $\mathbf{Z}\setminus\{1,-1\}$. So it will be sufficient to establish that the constant term of $\operatorname{res}(Q_m,Q_n)$ is 1 or -1. Since this constant term is obtained by letting '$k=0$' in Q_m and Q_n, we are led to define, for every $r\in\mathbf{N}$, $\varphi_r(X)=Q_r(X|k=0)\in\mathbf{Z}[X]$; it is of degree $r-1$ and also given by

$$\varphi_1=1,\ \varphi_2=X\ \text{ and } \varphi_{r+1}=X\varphi_r+\varphi_{r-1}\ \text{ if } r\geq 2.$$

(Hence the φ_r are so called "Fibonacci polynomials"; in particular $\varphi_r(1)$ is the usual Fibonacci number F_r.) Matricially

$$\begin{pmatrix}\varphi_{r+2} & \varphi_{r+1}\\ \varphi_{r+1} & \varphi_r\end{pmatrix}=\begin{pmatrix}X & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi_{r+1} & \varphi_r\\ \varphi_r & \varphi_{r-1}\end{pmatrix};$$

then, if we define $\varphi_0=0$,

$$\begin{pmatrix}\varphi_{r+2} & \varphi_{r+1}\\ \varphi_{r+1} & \varphi_r\end{pmatrix}=\begin{pmatrix}X & 1\\ 1 & 0\end{pmatrix}^{r+1}.$$

Therefore

$$\begin{pmatrix} \varphi_{r+s+2} & \varphi_{r+s+1} \\ \varphi_{r+s+1} & \varphi_{r+s} \end{pmatrix} = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^{r+(s+1)} = \begin{pmatrix} \varphi_{r+1} & \varphi_r \\ \varphi_r & \varphi_{r-1} \end{pmatrix} \begin{pmatrix} \varphi_{s+2} & \varphi_{s+1} \\ \varphi_{s+1} & \varphi_s \end{pmatrix};$$

so $\varphi_{r+s} = \varphi_{r-1}\varphi_s + \varphi_r\varphi_{s+1}$.

We first show by a recurrence on r that $\mathrm{res}(\varphi_{r+1},\varphi_r) = \pm 1$ for all $r \in \mathbf{N}$. Clearly $\mathrm{res}(\varphi_2,\varphi_1) = 1$. Suppose $\mathrm{res}(\varphi_r,\varphi_{r-1}) = \pm 1$; then $\mathrm{res}(\varphi_{r+1},\varphi_r) = \mathrm{res}(X\varphi_r + \varphi_{r-1},\varphi_r) = \pm\mathrm{res}(\varphi_{r-1},\varphi_r) = \pm\mathrm{res}(\varphi_r,\varphi_{r-1}) = \pm 1$ by (e) (the leading coefficient of φ_r being 1).

We next show that for all s, t in $\mathbf{N}$, there exist h_t^s in $\mathbf{Z}[X]$ with $\varphi_{ts} = h_t^s\varphi_s$. This is clear for $t = 1$. Suppose $t \geq 1$ and $\varphi_{ts} = h_t^s\varphi_s$ with h_t^s in $\mathbf{Z}[X]$; then $\varphi_{(t+1)s} = \varphi_{ts+s} = \varphi_{ts-1}\varphi_s + \varphi_{ts}\varphi_{s+1} = (\varphi_{ts-1} + h_t^s\varphi_{s+1})\varphi_s$.

Take now n, m in $\mathbf{N}$ with $n < m$ and write $m = qn + r$ with $0 \leq r < n$. We have $\varphi_m = \varphi_{qn+r} = \varphi_{qn-1}\varphi_r + \varphi_{qn}\varphi_{r+1}$; so by (e)

$$\mathrm{res}(\varphi_m,\varphi_n) = \mathrm{res}(\varphi_{qn-1}\varphi_r + h_q^n\varphi_n\varphi_{r+1},\varphi_n) = \pm\mathrm{res}(\varphi_{qn-1}\varphi_r,\varphi_n).$$

Now

$$\pm 1 = \mathrm{res}(\varphi_{qn},\varphi_{qn-1}) = \mathrm{res}(h_q^n\varphi_n,\varphi_{qn-1}) = \mathrm{res}(h_q^n,\varphi_{qn-1}) \cdot \mathrm{res}(\varphi_n,\varphi_{qn-1}).$$

But $\mathrm{res}(h_q^n,\varphi_{qn-1})$ and $\mathrm{res}(\varphi_n,\varphi_{qn-1})$ are both integers (by (a)); this implies that $\mathrm{res}(\varphi_n,\varphi_{qn-1}) = \pm 1$. Hence

$$\begin{aligned} \mathrm{res}(\varphi_m,\varphi_n) &= \mathrm{res}(\varphi_{qn+r},\varphi_n) = \pm\mathrm{res}(\varphi_{qn-1}\varphi_r,\varphi_n) \\ &= \pm\mathrm{res}(\varphi_{qn-1},\varphi_n) \cdot \mathrm{res}(\varphi_r,\varphi_n) = \pm\mathrm{res}(\varphi_r,\varphi_n). \end{aligned}$$

Let $r = r_1 > \ldots > r_l > r_{l+1} = 0$ be the sequence of remainders obtained by the Euclidean algorithm applied to (m,n). Iterating the above calculation we get:

$$\mathrm{res}(\varphi_m,\varphi_n) = \pm\mathrm{res}(\varphi_{r_1},\varphi_n) = \pm\mathrm{res}(\varphi_{r_1},\varphi_{r_2}) = \ldots = \pm\mathrm{res}(\varphi_{r_{l-1}},\varphi_{r_l}).$$

When m and n are relatively prime, $r_l = 1$; so $\varphi_{r_l} = 1$ and (b) gives $\mathrm{res}(\varphi_m,\varphi_n) = \pm 1$. The proposition is established.

Remarks. 1. When m and n are not relatively prime, $1 < r_l$ and $r_l | r_{l-1}$; so $\deg\varphi_{r_l} > 1$, $\varphi_{r_l} | \varphi_{r_{l-1}}$ and $\mathrm{res}(\varphi_m,\varphi_n) = 0$. **2.** The end of the proof is inspired by the presentation of Fibonacci numbers in [5, I.11]. **3.** Define a function f on $\mathcal{A}_k$ so: take a vertex x of $\mathcal{A}_k$ and set $f(y) = (-1)^{d(x,y)}$; f is not harmonic but $\mathcal{M}_{2r}f = f$ for all $r \in \mathbf{Z}_+$.

Commentary. For a degree $k > 2$, the condition $\gcd(m,n) = 1$ is *not* necessary; for example $\mathrm{res}(Q_6,Q_3) = k^2(k-1)^4(k-2)^2$. In fact, computing $\mathrm{res}(Q_m,Q_n)$ for small values of m and n ($n < m$, $3 \leq n \leq 10$, $6 \leq m \leq 25$) one gets the impression that, for $k > 2$, a plausible necessary and sufficient condition is the one in remark 3: m

and n not both even! However, the constant term $k(k-1)^{r-1}$ in the recurrence relation ($*$) defining the polynomials Q_{r+1} makes the full use of the properties of the resultant ((e) in particular) difficult. By letting $k=0$, which has no meaning geometrically (empty tree?), we get a much handier recurrence; the condition we obtain is the best possible with this reduction, as shown by remark 1.

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